Studying the Liberation Points of the Sun-Earth-Moon System

A.H. Ibrahim, M.N. Ismail, K.H.I. Khalil

Abstract— the restricted four-body problem is formulated; liberation points for the system are obtained and compared with those of the three-body problem. A MATHEMATICA code was constructed to compute trajectories for the positioning of spacecraft at any liberation points by numerical integration, using the Runge-Kutta forth order method. The Sun-Earth-Moon model is used as the main system, with the spacecraft as the fourth body. Gravitational forces are considered, while zero velocity curves for the restricted three-body (Earth-Moon-Spacecraft) and restricted four-body (Sun-Earth-Moon-Spacecraft) problems are obtained and plotted.

Key Words— Liberation points, Runge-Kutta, Sun-Earth-Moon model, zero velocity curves

1. INTRODUCTION

The four-body problem has been treated as one of the important objects of celestial mechanics. Work on this issue commenced at the end of the nineteenth century when Hill was the first to discover a periodic solution to the n-body problem. Poincare, in developing a method of treating the three-body problem, found an infinite number of periodic solutions. The four-body problem was also tackled by Moulton [15], who happened upon many particular solutions for the motion of the three-body problem, such as Jacobi integrals and liberation points. Tapley [18] studied the motion of a space vehicle positioned at L4 or L5 of the Earth's moon, using a model that included the perturbing effects of radiation pressure and the gravitational attraction of the Sun. He also invoked Runge-Kutta integration to obtain the trajectories of spacecraft around liberation points. The existence of stable, periodic orbits around the triangular point of the Earth-Moon system, which is perturbed by the Sun, was determined by Schechter and Kolenkiewicz [17,14], who calculated a periodic solution for the Sun-Earth-Moon system using numerical computations. In another study, Farquhar [9] studied the restricted four-body problem (Sun, Earth, Moon, satellite), finding that the liberation point L2 shifted by 295.10 km from L2, while for the Earth-Moon system, Guzman [12] found that L2 shifted by 300 km, which is the same result as that obtained in this work. Brouke [5] used numerical integration with the Runge-Kutta method to obtain the periodic orbits for the general three-body problem. There are several types of four-body model, which have been studied by a variety of authors; for example, Pernicka, Bell, Guzman and others [16,4,12] used the Sun-Earth-Moon model to study the four-body problem. In this model, the Sun, Earth, and Moons are treated as distinct particles of finite mass. Cronin, Gomez, Jorba, and Gabern [8,10, 11,13] studied the four-body problem using the bi-circular model, which considers the Earth and the Moon as two primaries revolving in circular orbits around their barycenter, while the Sun is considered as a third body moving in a circular orbit around the central masses of the Earth-Moon-Sun system; the three primaries move in the same plane. Meanwhile, Andreu [1] studied the quasi-bi-circular model in his PhD thesis. Baltagiannis, Papadakis, Burgos, and Delgado [2,3,6,7] studied the restricted four-body problem as equal masses located at the vertices of an equilateral triangle, which attain equilibrium points, zero velocity curves, and families of periodic orbits. In this paper, the Sun-Earth-Moon model is used to tackle the four-body problem.

2. RESTRICTED FOUR-BODY PROBLEM

To obtain the liberation points of the four-body problem, the Sun-Earth-Moon-spacecraft system is considered as follows: MS is the mass of the Sun, ME is the mass of the Earth, MM is the mass of the Moon, while m is the infinitesimal mass of the spacecraft, and CM is the center of the mass of the Earth and Moon. The Sun, Earth, and Moon all rotate around the barycenter of the entirety of system B. The spacecraft moves near the Earth-Moon system.

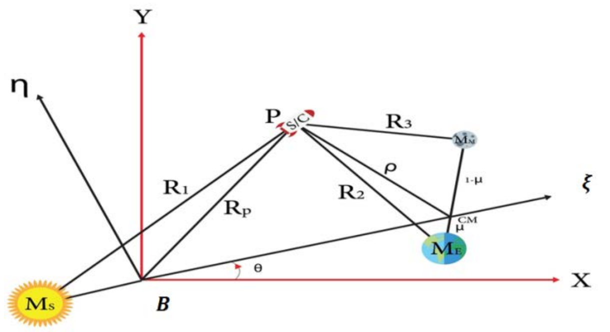

Figure 1The Earth-Moon-Sun Configuration in Rotating Coordinates

Figure 1 shows inertial and synodicalframes for Sun-Earth-Moon-spacecraft system. $x, y,$ and z is the inertial coordinates for the system, with the origin B at the mass center of the four-body system. ξ, η and ζFrame rotate with angular velocity Ω about the barycenter B which$\theta =$

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$\omega\, t$. The equations of motion in the inertial coordinates become:

$$\ddot{x} = \frac{-GM_S\,(x - x_s)}{[(x - x_S)^2 + (y - y_S)^2 + (z - z_S)^2]^{\frac{3}{2}}} - \frac{-GM_E(x - x_E)}{[(x - x_E)^2 + (y - y_E)^2 + (z - z_E)^2]^{\frac{3}{2}}} - \frac{-GM_M(x - x_M)}{[(x - x_M)^2 + (y - y_M)^2 + (z - z_M)^2]^{\frac{3}{2}}} \quad (1)$$

$$\ddot{y} = \frac{-GM_S\,(y - y_S)}{[(x - x_S)^2 + (y - y_S)^2 + (z - z_S)^2]^{\frac{3}{2}}} - \frac{-GM_E(y - y_E)}{[(x - x_E)^2 + (y - y_E)^2 + (z - z_E)^2]^{\frac{3}{2}}} - \frac{-GM_M(y - y_M)}{[(x - x_M)^2 + (y - y_M)^2 + (z - z_M)^2]^{\frac{3}{2}}} \quad (2)$$

$$\ddot{z} = \frac{-GM_S\,(z - z_S)}{[(x - x_S)^2 + (y - y_S)^2 + (z - z_S)^2]^{\frac{3}{2}}} - \frac{-GM_E(z - z_E)}{[(x - x_E)^2 + (y - y_E)^2 + (z - z_E)^2]^{\frac{3}{2}}} - \frac{-GM_M(z - z_M)}{[(x - x_M)^2 + (y - y_M)^2 + (z - z_M)^2]^{\frac{3}{2}}} \quad (3)$$

The canonical units of masses and distances are used, dividing all masses by the total mass of the Earth-Moon and dividing all distances by that between the Earth and Moon, so that the gravitational constant and mean motion of the Earth-Moon are taken as being in unity.

The masses and distances of the Sun, Earth, and Moon are: Mass of the Sun, MS = 1.99 * 10 ^ 30 kg; Mass of the Earth, ME = 5.98 * 10 ^ 24 kg; Mass of the Moon, MM = 7.35 * 10 ^ 22 kg; Earth-Moon distance, d1 = 3.844 * 10 ^ 5 km; Earth-Sun distance, d2 = 1.496 * 10 ^ 8 km.Then, the masses of the Earth, Moon, and Sun in the canonical system are: Mass of the Earth $= \mu_E = 1 - \mu = \frac{ME}{ME + MM} = 0.9878715$; Mass of the Moon $= \mu_M = \mu = \frac{ME}{ME + MM} = 0.0121506683$; Mass of the Sun $= \mu_S = \frac{MS}{ME + MM} = 328900.48$; the distance between the Sun and the center of the system = Rs = 389.1723985.

The coordinates of the Earth, Moon, and Sun, with respect to B, are given by:

$x_E = \mu \cos t, \quad y_E = \mu \sin t, \quad z_E = 0;$

$x_M = (\mu - 1)\cos t, \quad y_M = (\mu - 1)\sin t, \quad z_M = 0;$

$x_S = R_S \cos\theta, \quad y_S = R_S \sin\theta, \quad z_S = 0.$

The distances of the spacecraft from the Sun, Earth, and Moon, in rotating coordinates, are given as:

$R_1 = \sqrt{(\xi - \xi_S)^2 + (\eta - \eta_S)^2 + \zeta^2}$;
$R_2 = \sqrt{(\xi + \mu)^2 + \eta^2 + \zeta^2}$;

$R_3 = \sqrt{(\xi - \mu + 1)^2 + \eta^2 + \zeta^2}$

Now, the transformation between the $x, y,$ and z frames and the $\xi, \eta,$ and ζ frames can be represented by:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \xi \\ \eta \\ \zeta \end{bmatrix}$$

The equations of motion of the spacecraft in the rotating system become:

$$\ddot{\xi} - 2\dot{\eta} - \xi = -\frac{(\xi-\mu)(1-\mu)}{{R_2}^3} - \frac{(\xi-1-\mu)\mu}{{R_3}^3} - \frac{(\xi - R_S\cos\theta)\mu_S}{{R_1}^3}, \quad (4)$$

$$\ddot{\eta} + 2\dot{\xi} - \eta = -\frac{\mu}{{R_2}^3}\eta - \frac{(1-\mu)\eta}{{R_3}^3} - \frac{(\eta - R_S\sin\theta)\mu_S}{{R_1}^3}, \quad (5)$$

$$\ddot{\zeta} = -\frac{\zeta\mu}{{R_2}^3} - \frac{\zeta(1-\mu)}{{R_3}^3} - \frac{\zeta\mu_S}{{R_1}^3} \quad (6)$$

3. JACOBI INTEGRAL

The last equations could be put in these formulae:

$$\ddot{\xi} - 2\dot{\eta} = \frac{\partial U}{\partial \xi}, \quad (7)$$

$$\ddot{\eta} + 2\dot{\xi} = \frac{\partial U}{\partial \eta}, \quad (8)$$

$$\ddot{\zeta} = \frac{\partial U}{\partial \zeta} \quad (9)$$

Where $\quad U = \frac{1-\mu}{R_{\mathbb{S}E}} + \frac{\mu}{R_{\mathbb{S}M}} + \frac{\mu_S}{R_{\mathbb{S}S}} + \frac{\xi^2 + \eta^2}{2}$

is the potential in the rotating coordinate system; by multiplying (7), (8), and (9) by $\dot{\xi}, \dot{\eta}$ and $\dot{\zeta}$

respectively, and adding the equation, this becomes:

$$\ddot{\xi}\dot{\xi} + \ddot{\eta}\dot{\eta} + \ddot{\zeta}\dot{\zeta} = \frac{\partial U}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial U}{\partial \eta}\frac{\partial \eta}{\partial t} + \frac{\partial U}{\partial \zeta}\frac{\partial \zeta}{\partial t}, \quad (10)$$

$$\frac{1}{2}\frac{d}{dt}\left[\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right] = \frac{dU}{dt} \quad (11)$$

$$\left[\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2\right] = 2U - C \quad (12)$$

This equation is called the Jacobi integer; when velocity equals zero, equation (12) becomes

$$2U = C \tag{13}$$

which we can obtain from this equation on zero velocity curves. Figures (2) and (3) display the counterplots for the Sun-Earth-Moon and Earth-Moon systems.

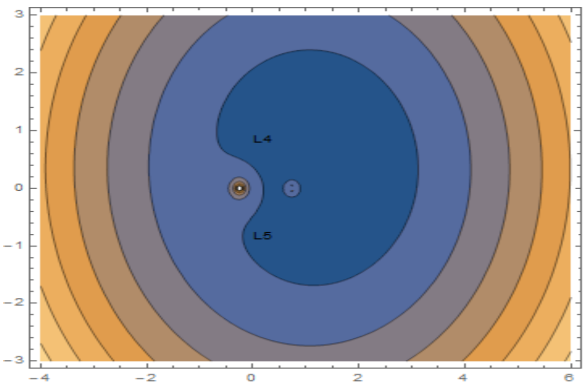

Figure 2 Counterplot for Sun-Earth-Moon system

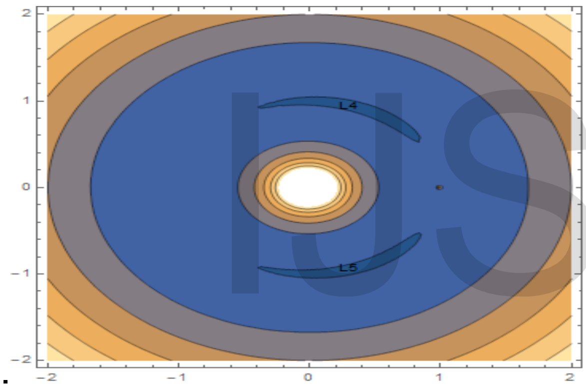

Figure 3 Counterplot for Earth-Moon system.

4. Liberation points for restricted four-body problem

The condition of equilibrium pints is deduced when making the velocities' and accelerations' components equal to zero is given by equations (4), (5), and (6).

$$\xi = -\frac{(\xi-\mu)(1-\mu)}{R_{sE}{}^3} - \frac{(\xi-1-\mu)\mu}{R_{sM}{}^3} - \frac{(\xi - R_S\cos\theta)\mu_S}{R_{sS}{}^3}, \tag{14}$$

$$\eta = -\frac{\mu}{R_{sE}{}^3}\eta - \frac{(1-\mu)\eta}{R_{sM}{}^3} - \frac{(\eta - R_S\sin\theta)\mu_S}{R_{sS}{}^3}, \tag{15}$$

$$0 = -\frac{\zeta\mu}{R_{sE}{}^3} - \frac{\zeta(1-\mu)}{R_{sM}{}^3} - \frac{\zeta\mu_S}{R_{sS}{}^3} \tag{16}$$

To determine the trajectories of the spacecraft near the liberation points, a code was constructed using the MATHEMATICA program to solve the equations of motion of the spacecraft at the liberation points by numerical integration using the Runge-Kutta forth order method.

Table 1 displays the non-dimensional kilometers of distances for the liberation points of the Earth-Moon system; the liberation points after the effect of the Sun are considered. Note that in the table, the shift on L2 has the same value as that attained by Guzman (2001).

Table 1 Liberation points for Earth-Moon system and Sun-Earth-Moon system

Liberation points	Earth-Moon Non-dimensional (km)		Sun-Earth-Moon Non-dimensional (km)		Shift Km
L1	0.83692	32171	0.83641	32152	194
L2	1.1556821	444244	1.1549	443944	300
L3	-1.005062	-386346	-1.0069	-38705	706
L4	(0.83692, 0.99387)	(32171, 382043)	(0.8364, 0.9924)	(32151, 381479)	(198, 564)
L5	(0.83692, -.99387)	(32171,-38204)	(0.8364, -0.9924)	(32151, -38148)	(198, -564)

Figures 4-7 display comparisons between the trajectories of spacecraft near liberation points L1-L4 in the Earth-Moon and Sun-Earth-Moon systems. In Figure 4, the solid curves represent trajectories of spacecraft, starting at L1 in the Earth-Moon system, while the dashed curves represent trajectories of spacecraft near L1 for the Earth-Moon system after taking into account the effect of the Sun in calculations. Note that the dashed curves shift toward the left around the solid curves; the same phenomenon can be seen in Figure 5. In Figure 6, the motion of the spacecraft starts from an "at rest" position at the liberation points; at L3, we observe that the spacecraft is still at the rest point for period 60in the Earth-Moon model, but the far rest point is after period 16 when the effect of the Sun is considered.

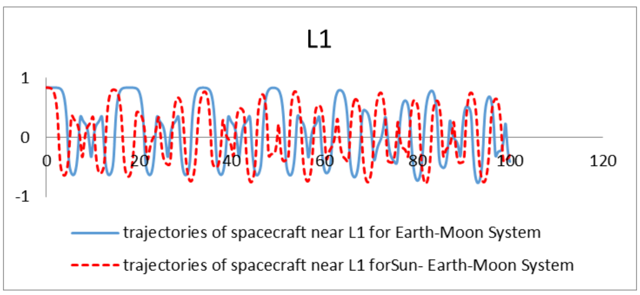

Figure 4 Trajectories of spacecraft near L1 for two systems

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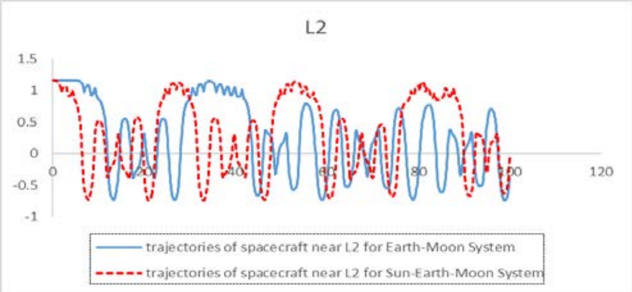

Figure 5 Trajectories of spacecraft near L2 for two systems

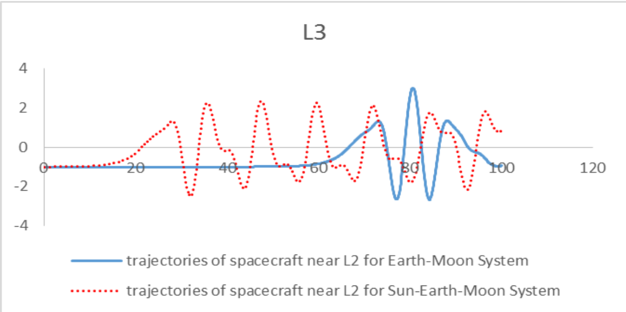

Figure 6Trajectories of spacecraft near L3 for two systems

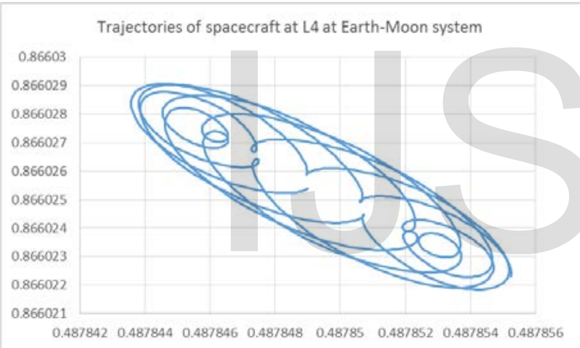

Figure 7 Trajectories of spacecraft near L4 for Earth-Moon system

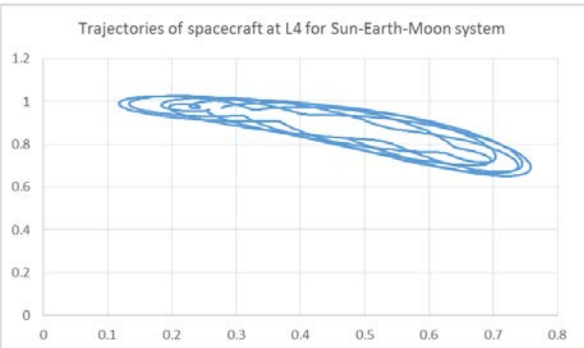

Figure 8 Trajectories of spacecraft near L4 for Sun-Earth-Moon system

5. Conclusion

In this study, the equations of motion for the restricted four-body problems were deduced. These equations are applicable to two systems: The Earth-Moon and Sun-Earth-Moon systems. The locations of the liberation points for the former were obtained, as were those of the latter, after taking the effect of the Sun into account. The shift between the two models was mentioned. Zero velocity curves were deduced and plotted for both systems, while the trajectories for the spacecraft near the liberation points of the Earth-Moon system were computed and plotted using the Sun-Earth-Moon system.

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